Journal of Statistical Physics, Vol. 92, Nos. 1/2, 1998

Time Asymptotics for Solutions of Vlasov–Poisson Equation in a Circle

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Received February 9, 1998; final April 9, 1998

We prove that there exists a class of solutions of the nonlinear Vlasov–Poisson equation (VPE) on a circle that converges weakly, as $t \to \infty$, to a stationary homogeneous solution of VPE. This behavior is called, in the linear case, Landau damping. The result is obtained by constructing a suitable scattering problem for the solutions of the Vlasov–Poisson problem. A consequence of this result is that a class of stationary solutions of the Vlasov–Poisson equation is unstable in a weak topology.

KEY WORDS: Vlasov–Poisson equations; scattering theory; asymptotic behavior of solutions.

1. INTRODUCTION

It is well known[15, 28] that the motion of a plasma of electrons in a uniform background of ions can be described, in the collisionless case, by the 1D Vlasov–Poisson equations

$$\begin{gathered} \partial_t f(x, v, t) + v\, \partial_x f(x, v, t) + E(x, t)\, \partial_v f(x, v, t) = 0 \\ \partial_x E = \rho(x, t) - \rho_0 \\ \rho(x, t) = \int_{\mathbb{R}} f(x, v, t)\, dv. \end{gathered} \tag{1.1}$$

In (1.1) $f(x, v, t)$ represents the density of electrons at location x, traveling with velocity v at time t, ρ is the space density and ρ_0 is the density of ions (assumed constant) added to make the system neutral. Notice that by $E(x, t)$ we mean the force field, which is minus the electric field.

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0022-4715/98/0700-0301$15.00/0

Let us recall[20] that a simple steady solution of (1.1) is provided by the pair

$$f(x, v, t) = h(v), \qquad E = 0, \qquad \text{with } \int_{\mathbb{R}} h(v)\, dv = \rho_0$$

In a classical paper of 1946[14, 15] Landau considered the linear Vlasov–Poisson equation (i.e., the Vlasov–Poisson equation linearized around a stationary homogeneous solution) and showed that the perturbation is asymptotically damped, and the electrical field vanishes, as $t \to \infty$.

This linear phenomenon by which a perturbed plasma relaxes toward a homogeneous equilibrium is usually called Landau damping. A complete description of Landau damping in the analytic framework is presented in ref. 18.

In this paper we show that there exists a suitable class of solutions of the full problem (1.1) on a circle, which relaxes, asymptotically, to a homogeneous equilibrium state. This is, as far as we know, the first rigorous proof of damping on a bounded set, namely not due to dispersive effects.

However, our analysis is far from complete: indeed, we are not able to characterize fully the class of initial data exhibiting such behavior.

The idea of the proof is, roughly, the following.

If f becomes homogeneous, and therefore the electrical field vanishes, then, for large t, we expect that f behaves as the free evolute of some suitable phase space density f^*:

$$f(x, v, t) \simeq f^*(x - vt, v) \tag{1.2}$$

Thus, instead of solving VPE with an initial datum and trying to understand whether or not the solution becomes homogeneous, we give an asymptotic datum f^* and try to solve VPE with the condition

$$\|f(x, v, t) - f^*(x - vt, v)\|_{L_\infty(x, v)} \to 0 \qquad \text{as} \quad t \to \infty \tag{1.3}$$

In particular, as we shall see, we try to construct a solution by perturbing around $f^*(x - vt, v)$.

This program may be performed if f^* is sufficiently smooth. In particular we shall require sufficient conditions on f^* to assure an exponential decay of the electrical field as $t \to \infty$.

Once a solution of the Vlasov–Poisson equation satisfying (1.3) is found, it is easy to prove also the homogeneous nature of the result. In fact

(1.3) says that f behaves asymptotically as the free evolution of f^* and the free evolute of the density become homogeneous, i.e.,

$$\lim_{t\to\infty} f(x, v, t) = \lim_{t\to\infty} f^*(x - vt, v) = h(v) \tag{1.4}$$

where $h(v) = (1/2\pi)\int_{S^1} f^*(x, v)\, dx$ is the spatial mean of f^*. Here the convergence is in the sense of the weak convergence of the measures.

This kind of construction is typical in scattering theory (see, for example, ref. 23), that is, we construct a solution of the Vlasov–Poisson equation that has a given asymptotic behavior. In this language the solution of the full asymptotic problem would be to prove the asymptotic completeness.

In Section 2 we state the problem and give some general definitions. In particular we define the Vlasov–Poisson equation on a circle with an asymptotic condition.

We prove that the free motion is homogeneous: this is a well-known result; see for instance, ref. 8.

In Section 3 we prove the main result of this paper. Given f^* sufficiently smooth, we construct, by iteration, a solution of VPE that satisfies the asymptotic behavior (1.3). The main tool to prove this result is a contraction argument (Lemma 3.1 below) that we use to construct a sequence of linear evolution problems converging to a solution of the VPE.

Finally, in Section 4, we prove the instability, in the weak topology, for a class of stationary solutions of VPE. This class is explicitly characterized. In particular it is possible to prove that a Maxwellian $\exp(-av^2)$, is unstable if a is sufficiently large.

We conclude this introduction with some general remarks.

The existence problem and the qualitative properties of the VPE have been extensively investigated. Here we quote some references (without claiming to be complete) and address the reader to ref. 3 for an excellent and extensive review. Existence of smooth solutions of the Vlasov–Poisson equation in dimension 1 was obtained by Iordanskii[13], and in dimension 2 by Ukai and Okabe[26]. In dimension 3 global existence was obtained by Pfaffelmoser[21] and then his result was simplified by Shaeffer[24]. In dimension 1 the existence of measures-valued solutions was obtained by Zheng and Majda,[28] while Majda *et al.*[16] showed nonuniqueness of solutions of the Vlasov–Poisson equation for singular initial data. Time decay of the solutions of the Vlasov–Poisson equation on the whole space has been characterized by Glassey and Shaeffer[7] (in the linearized case), Ilner and Rein,[12] and Perthame.[22]

Dispersive arguments were used by Bardos and Degond[2] to prove that solutions of the 3D Vlasov–Poisson equations (in all $\mathbb{R}^3$), with small initial data, asymptotically decay. Scattering theory has also been used to investigate the long-time behavior of solutions of nonlinear wave equations (nonlinear Schrödinger equation, Klein–Gordon equation).[10, 11, 25]

The problem we study here differs from those in the above references because we work in a compact configuration space (the circle).

It is well known that there is an analogy between the 1D Vlasov–Poisson equation and the 2D Euler equation expressed in terms of vorticity. The result in this paper suggests that, following the same approach, one could try to study the homogenization problem for the Euler equation solutions. This question was considered in ref. 5, where the asymptotic behavior of a particular vortex patch was studied. More precisely, in ref. 5 it was shown that homogenization is realized for the simplified model of the characteristic function of a set $D = C \cup A$, where C is a circular domain and A is an annulus with regular boundary, which evolves passively (that is, A evolves only under the action of the velocity vector field due to C).

Finally, a statistical mechanics approach to the study of the long-time behavior of the Vlasov–Poisson equation was proposed in ref. 19.

2. GENERAL FACTS AND NOTATIONS

On the phase space $\Omega \equiv S^1 \times R$ we consider the following norm: if $z = (x, v)$, then $|z| \equiv |x - x'|_{S^1} + |v - v'|$, where $|x - x'|_{S^1}$ is the distance on the circle:

$$|x - x'|_{S^1} = \min_{k \in \mathbb{Z}} |x - x' + k|$$

In the following, when no confusion can occur, we omit the index S^1 for notational simplicity.

In the following we shall often use the notation

$$E(x, t) = \int_{S^1} dy\, B(x - y)\, \rho(y, t) \tag{2.1}$$

where $B(x)$ solves $\partial_{xB} = \delta(x) - 1/(2\pi)$ on S^1 and therefore

$$\begin{aligned} B(x) &= \frac{1}{2} - \frac{x}{2\pi} \qquad \text{for} \quad x \in [0, 2\pi) \\ B(x + 2\pi) &= B(x). \end{aligned} \tag{2.2}$$

Sometimes it is convenient to think of B as extended periodically in the whole line. In this case note that B is discontinuous at $2k\pi$, $k \in \mathbb{Z}$. Moreover,

$$|B(x)| \leqslant 1/2 \tag{2.3}$$

and if the interval $[x, y]$ does not contain $2k\pi$ for any $k \in \mathbb{Z}$, then

$$|B(x) - B(y)| \leqslant \frac{1}{2\pi} |x - y| \tag{2.4}$$

In this paper we look for solutions of the VPE on a circle,

$$\begin{gathered} \partial_t f(x, v, t) + v\, \partial_x f(x, v, t) + E(x, t)\, \partial_v f(x, v, t) = 0 \\ \partial_x E = \rho(x, t) - \rho_0 \\ \rho(x, t) = \int_{\mathbf{R}} f(x, v, t)\, dv, \end{gathered} \tag{2.5}$$

which satisfy the asymptotic condition

$$\| f(x, v, t) - f^*(x - vt, v) \|_{L_\infty(x, v)} \to 0 \qquad \text{as} \quad t \to \infty \tag{2.6}$$

for a given f^* in a suitable class to be defined later.

More precisely we shall consider the following problem.

Definition 2.1. We say that the triple (Φ_t, E, f) is a solution of the VPE if for any $(x, v) \in \Omega$ and $t \geqslant t_0$, $\Phi_t(x, v) \equiv (X(x, v, t), V(x, v, t))$ satisfies

$$\begin{gathered} \dot{X} = V \\ \dot{V} = E(X, t) \\ \lim_{t \to \infty} X - Vt = x \\ \lim_{t \to \infty} V = v \end{gathered} \tag{2.7}$$

where

$$E(x, t) \equiv \int_\Omega dy\, dv\, B(x - y)\, f(y, v, t), \tag{2.8}$$

and

$$f(x, v, t) = f^* \circ (\Phi_t)^{-1}(x, v) \tag{2.9}$$

Note that solutions of problem (2.7)–(2.9) are weak solutions of the problem (2.5). We look for solutions of (2.7)–(2.9) which also satisfy condition (2.6) and this will follow automatically under suitable smoothness conditions.

Definition 2.2. (Homogenization). A solution $f(x, v, t)$ of the VPE becomes homogeneous if there exists a function $h \in L_1(v)$ such that

$$\lim_{t \to \infty} f(x, v, t) = h(v) \tag{2.10}$$

where the limit is understood in the sense of the weak convergence of the measures.

It is easy to prove that the free motion homogenizes.

Theorem 2.1. If f solves $\partial_t f + v\, \partial_x f = 0$, $f(x, v, 0) = f_0(x, v)$, where $f_0 \in L_1 \cap L_\infty$, i.e.,

$$f(x, v, t) = f_0(x - vt, v)$$

then there exists the weak limit

$$\lim_{t \to \infty} f(x, v, t) = h(v)$$

where

$$h(v) = \frac{1}{2\pi} \int_{S^1} dx\, f_0(x, v)$$

is the spatial average of f_0.

Proof. Given a test function $\phi(z)$, $z = (x, v)$, $\phi \in C^0$, with a compact support, let us consider $\langle \phi(z), f(z, t) \rangle$, the scalar product in L_2 of ϕ and f. We have

$$\langle \phi(z), f(z, t) \rangle = \langle \phi(z), f_0(x - vt, v) \rangle = \langle \phi(x + vt, v), f_0(z) \rangle$$

Let us call $\hat{g}(a, b)$ the Fourier transform of $g(x, v)$ with respect to x and v, i.e.,

$$\hat{g}(a, b) = \frac{1}{2\pi} \int_{\Omega} dx\, dv\, e^{-iax - ibv} g(x, v)$$

We can notice that $\hat{\phi}(x + vt, v)(a, b) = \hat{\phi}(a, b + at)$.

From the Parceval equality it follows that

$$\begin{aligned}&\langle \phi(x+vt, v), f_0(z)\rangle\\&\quad=\langle \hat{\phi}(a, b+at)\,\hat{f}_0(a, b)\rangle\\&\quad=\int_{\mathbf{R}}[\bar{\hat{\phi}}(0, b)\,\hat{f}_0(0, b)]\,db+\sum_{a\neq 0}\int_{\mathbf{R}}[\bar{\hat{\phi}}(a, b+at)\,\hat{f}_0(a, b)]\,db,\end{aligned}$$

where $\bar{\hat{\phi}}$ is the complex conjugate of $\hat{\phi}$. The first term in the sum exactly gives $(1/2\pi)\int_{\Omega}\phi(x, v)\,h(v)\,dx\,dv$. If one applies to the second term in the sum the Lebesgue dominated convergence theorem, the result follows.

3. THE MAIN RESULT

The procedure we follow to study the problem (2.5)–(2.6) (in the sense of Definition 2.1) is iterative.

More precisely, given f^*, let us define, for $(x, v, t)\in\Omega\times[t_0, \infty)$ and $n=0, 1, \ldots$, the sequence of linear problems

$$\partial_t f_{n+1}(x, v, t)+v\,\partial_x f_{n+1}(x, v, t)+E_n(x, t)\,\partial_v f_{n+1}(x, v, t)=0, \tag{3.1}$$

to be solved with the asymptotic condition

$$\|f_{n+1}(x, v, t)-f^*(x-vt, v)\|_{L_\infty(x, v)}\to 0 \qquad \text{as} \quad t\to\infty \tag{3.2}$$

where

$$E_0=0$$

and, for $n>0$,

$$\begin{aligned}\partial_x E_n&=\rho_n(x, t)-\rho_0\\ \rho_n(x, t)&=\int_{\mathbf{R}} f_n(x, v, t)\,dv.\end{aligned} \tag{3.3}$$

Let us notice that for $n=0$ the evolution is just the free one:

$$f_1(x, v, t)=f^*(x-vt, v)$$

In the following we prove that, if f^* belongs to a suitable set, the problem (3.1)–(3.2), with the condition (3.3), can be solved for all n. Moreover, $f_n(x, v, t)$ converges, as $n\to\infty$, to a solution $f(x, v, t)$ of the problem (2.5)–(2.6).

We anticipate that the main ingredient of the proof is the contractivity of the operator $\mathscr{F}$ (see the definition below) that applied to E_n gives E_{n+1} in a suitable norm (see Definition 3.2). The contractivity properties of this operator are proved in Lemma 3.1, while the main result, which is a simple consequence of this lemma, is given in Theorem 3.2.

Definition 3.1. Given the positive constants a, a_1, a_2, we say that $f^* \in S_{a, a_1, a_2}$, if $f^* \geqslant 0$, and (i)

$$|\hat{f}^*(k_x, k_v)| \leqslant \frac{a_1}{1+k_x^2} e^{-a|k_v|}$$

where

$$\hat{f}^*(k_x, k_v) \equiv \frac{1}{2\pi} \int_{S^1} dx \int_{\mathbb{R}} dv\, e^{i(k_x x + k_v v)} f^*(x, v)$$

is the Fourier transform of f^* with respect to space and velocity, and (ii)

$$f^*(x, v) \leqslant \frac{a_2}{1+v^4}$$

Definition 3.2. Given $F(x, t)$, let us define $\|F\|_{a, t_0} = \sup_{t \geqslant t_0} e^{at} \|F(\cdot, t)\|_{L_\infty(S^1)}$.

Lemma 3.1. Let $f^* \in S_{a, a_1, a_2}$, where $a \geqslant 15\sqrt{a_2}$. Let $t_0 \geqslant 0$ and $F(x, t) \in C(S^1 \times [t_0, \infty))$ a given field, such that (i) $\|F\|_{a, t_0} e^{-at_0} \leqslant a$, and (ii) for any $t \geqslant t_0$, there exists $L_F \leqslant 24 a_2$ such that

$$|F(x, t) - F(y, t)| \leqslant L_F |x - y|$$

Then:

1. For any $(x, v) \in \Omega$, $t \geqslant t_0$, there exists a unique solution

$$\Phi_t(x, v) \equiv (X(x, v, t), V(x, v, t))$$

of

$$\begin{aligned} \dot{X} &= V \\ \dot{V} &= F(X, t) \\ \lim_{t \to \infty} X - Vt &= x \\ \lim_{t \to \infty} V &= v \end{aligned} \tag{3.4}$$

2. Defining f by

$$f(x, v, t) = f^* \circ (\Phi_t)^{-1}(x, v)$$

it turns out that f is a weak solution of

$$\partial_t f + v\,\partial_x f + F(x, t)\,\partial_v f(x, v, t) = 0$$

with the condition

$$\lim_{t \to \infty} \|f(x, v, t) - f^*(x - vt, v)\|_{L_\infty} = 0$$

3. Defining $\mathscr{F}(F)$ by

$$\mathscr{F}(F)(x, t) = \int_{S^1} dy\, B(x-y)\,\rho(y, t), \qquad \rho(x, t) = \int_{\mathbb{R}} dv\, f(x, v, t)$$

then

$$\|\mathscr{F}(F)\|_{a, t_0} \leqslant 4a_1 a_2 + \tfrac{1}{2}\|F\|_{a, t_0} \tag{3.5}$$

$$|\mathscr{F}(F)(x, t) - \mathscr{F}(F)(x', t)| \leqslant 24a_2\,|x - x'| \tag{3.6}$$

$$\|\mathscr{F}(F_1) - \mathscr{F}(F_2)\|_{a, t_0} \leqslant \tfrac{1}{2}\|F_1 - F_2\|_{a, t_0} \tag{3.7}$$

Moreover, as $F = 0$, $\mathscr{F}(0)$ satisfies

$$\|\mathscr{F}(0)\|_{a, t_0} \leqslant 4a_1 a_2 \tag{3.8}$$

The proof of this lemma is given in the Appendix.

Remark. Let us note that the trajectories in (3.4) are labeled with their asymptotic behavior instead of, as usual, with their initial condition.

Theorem 3.2. Assume that $f^* \in S_{a, a_1, a_2}$, where $a \geqslant 15\sqrt{a_2}$, and let $t_0 \geqslant \max(0, (1/a)\log 8a_1 a_2)$. Then, as $t \geqslant t_0$, there exists a triple (Φ_t, E, f) which satisfies (2.7)–(2.9). Moreover, f is a weak solution of VPE (2.5) which satisfies the asymptotic condition (2.6) and

$$\|E\|_{a, t_0} \leqslant 8a_1 a_2 \tag{3.9}$$

Proof. The proof, by induction, is a direct application of Lemma 3.1.

Let us consider the problem (3.1)–(3.3). We shall prove that, as $n = 1, 2, \ldots,$

$$\|E_n\|_{a,t_0} \leqslant 8a_1a_2 \tag{3.10}$$

$$|E_n(x,t) - E_n(x',t)| \leqslant 24a_2\,|x-x'| \tag{3.11}$$

$$\|E_{n+1} - E_n\|_{a,t_0} \leqslant \frac{1}{2^n}\,8a_1a_2 \tag{3.12}$$

Step $n = 1$. By (3.1)–(3.3) and the definition of $\mathscr{F}$ we have

$$\mathscr{F}(0) = E_1(x,t) = \int_\Omega dy\,dv\,B(x-y)\,f^*(y-vt, v)$$

by (3.8)

$$\|E_1\|_{a,t_0} \leqslant 4a_1a_2 \tag{3.13}$$

while by (3.6) it holds that

$$|E_1(x,t) - E_1(x',t)| \leqslant 24a_2\,|x-x'| \tag{3.14}$$

Therefore E_1 satisfies (3.10) and (3.11).

Step $n \Rightarrow n+1$. If E_n satisfies (3.10), (3.11), then E_n satisfies the hypothesis of Lemma 3.1. Therefore, by (3.5), as $E_{n+1} = \mathscr{F}(E_n)$, we get

$$\|E_{n+1}\|_{a,t_0} \leqslant 4a_1a_2 + \tfrac{1}{2}\,\|E_n\|_{a,t_0} \leqslant 8a_1a_2$$

which implies that E_{n+1} satisfies (3.10). Moreover, (3.6) implies that E_{n+1} satisfies (3.11), and (3.12) is a consequence of (3.7).

Finally, by (3.12) the sequence E_n is convergent to a Lipschitz function E which satisfies (3.10), (3.11). Then by Lemma 3.1 we get the existence of the solution.

Remark. The solution can be extended at time $t < t_0$ by applying an existence theorem for the Vlasov–Poisson equation; see, for example, ref. 13.

The initial datum is not explicitly characterized, but we can construct it iteratively; in fact, $f(x, v, t)$ is the limit of $f_n(x, v, t)$ and f_n converges to f in an exponential way.

Corollary 3.3. The solution $f(x, v, t)$ constructed in Theorem 3.2 becomes homogeneous.

Proof. In fact, for any $\varphi \in C_0(\Omega)$ we have

$$\int_\Omega dx\, dv\, \varphi(x, v)\, f(x, v, t)$$

$$= \int_\Omega dx\, dv\, \varphi(x, v)[f(x, v, t) - f^*(x - vt, v)]$$

$$+ \int_\Omega dx\, dv\, \varphi(x, v)\, f^*(x - vt, v)$$

The first term vanishes as $t \to \infty$ because $\|f(x, v, t) - f^*(x - vt, v)\|_{L_\infty(x, v)} \to 0$ and the second, because of Theorem 2.1, converges to $\int \varphi h$, where $h(v) \equiv (1/2\pi) \int_{S^1} dx\, f^*(x, v)$.

The solution we have found is not classical because f^* is only Holder continuous as a function of x. By requiring additional hypotheses on f^*, it is possible to obtain a classical solution.

Theorem 3.4 (Regularity). Assume that f^* satisfies the hypothesis of Theorem 3.2. Furthermore assume that $f^* \in C^1(\Omega)$, and that

$$|\nabla f^*(x, v)| \leqslant \frac{c}{1 + v^2} \tag{3.15}$$

where c is a positive constant. Then the solution $f(x, v, t)$ constructed in Theorem 3.2 is a classical (C^1) solution of (2.5)–(2.6). This theorem is proved in the Appendix.

4. INSTABILITY FOR A CLASS OF STATIONARY SOLUTIONS OF THE VPE AND SOME COMMENTS

It is well known (see, for example, refs. 4, 9, 17) that a stationary solution $h(v)$ of VPE is stable in L_1 if $h(v)$ has a finite kinetic energy, is a nonincreasing function as $v > 0$, and is a nondecreasing function as $v < 0$. In particular the Maxwellian $h(v) = e^{-av^2}$ $(a > 0)$ is a stationary stable solution of the VPE in the L_1 norm.

We note here that, for a certain class of these solutions, we have proved an instability result in a weak topology.

More precisely, we proved that if $f^*(x, v)$ belongs to the functional space S_{a, a_1, a_2} (for $a \geqslant 15\sqrt{a_2}$), then there exists a solution $f(x, v, t)$ of the VPE such that $\|f(x, v, t) - f^*(x - vt, v)\|_{L_\infty(x, v)} \to 0$ as $t \to \infty$. This fact implies, in particular, that $f(x, v, t)$ converges weakly, as $t \to \infty$, to $h(v) \equiv (1/2\pi) \int_{S^1} f^*(x, v)\, dx$.

Moreover, because of the time reversibility of the solutions of the VPE, we can construct an initial condition as close as we want (in a weak topology) to $h(-v)$ such that, after a certain time, its distance (in the same topology) from $h(-v)$ is greater than a certain fixed amount.

In other words, if $f^*(x, v)$ belongs to S_{a, a_1, a_2}, then $h(v)$ is an unstable stationary solution of the VPE [notice that if $f^*(x, v) \in S_{a, a_1, a_2}$ then $f^*(x, -v) \in S_{a, a_1, a_2}$].

In particular, we can prove instability (in the weak topology) for stable (in the L_1 norm) solutions of VPE. In the case of the Maxwellian, to find that $h(v) = e^{-av^2}$ is unstable, it is sufficient to take $f^*(x, v) = e^{-av^2}(1 + \lambda \cos x)$, with $|\lambda| < 1$ and $a > 0$ large enough, in order to satisfy the hypotheses of Theorem 3.2.

We conclude with a final remark. Our result proves the existence of solutions of VPE that converge weakly, as $t \to \infty$, to a steady solution $h(v)$. As was noticed by Landau, this phenomenon is of a different nature than relaxation at the equilibrium for collisional gases (i.e., for gases described by the Boltzmann equations). In particular, VPE are time-reversible equations, and the entropy $-\int f \log f$ is constant along the solutions. Nevertheless, one can notice that the entropy related to the steady solution h is larger than the entropy related to f. In fact, taking into account that the convergence is weak and the entropy functional is concave, one has

$$-\int h \log h \geqslant \limsup_{t \to \infty} \left(-\int f \log f \right)$$

The right-hand side is constant in time and equal to $-\int f^* \log f^*$, and the equality is realized only if $h = f$, that is, in the trivial case. Notice in fact that $h(v)$ is obtained by averaging $f^*(x, v)$ with respect to the x variable.

APPENDIX

Proof of Lemma 3.1.

Step 1. Let us consider the trajectories of the dynamical system

$$\dot{X} = V$$

$$\dot{V} = F(X, t)$$

Since F is Lipschitz and $\|F\|_{a, t_0}$ is bounded for any pair (x, v) in $S^1 \times \mathbb{R}$, there exists, uniquely, a trajectory $\Phi_t(z, v) \equiv (X(x, v, t), V(x, v, t))$ such that

$$\lim_{t \to \infty} X - Vt = x$$

$$\lim_{t \to \infty} V = v$$

In fact we can represent the solution as

$$\begin{aligned} X(x, v, t) &= x + vt + \int_t^\infty ds(s-t)\, F(X(s), s) \\ V(x, v, t) &= v - \int_t^\infty ds\, F(X(s), s) \end{aligned} \tag{A.1}$$

where the linear operator $P[X]$ defined as

$$P[X](t) = \int_t^\infty ds(s-t)\, F(X(s), s)$$

is contractive in the norm

$$\|X\| \equiv \sup_{t \geqslant t_0} e^{at}\, |X(t)|$$

In fact,

$$\begin{aligned} |P[X](t) - P[Y](t)| &\leqslant \int_t^\infty ds(s-t)\, |F(X(s), s) - F(Y(s), s)| \\ &\leqslant L_F \|X - Y\| \int_t^\infty ds(s-t)\, e^{as} \leqslant \frac{L_F}{a^2} e^{-at} \end{aligned}$$

which implies

$$\|P[X](t) - P[Y](t)\| \leqslant \frac{L_F}{a^2} \|X - Y\|$$

Once $X(x, v, t)$ has been obtained, $V(x, v, t)$ is given by the second of (A.1). In particular, it holds that

$$|V(x, v, t) - v| \leqslant \frac{\|F\|_{a, t_0}}{a} e^{-at} \leqslant 1 \tag{A.2}$$

Step 2. Fixed $t \geqslant t_0$, $\Phi_t(x, v) = (X(x, v, t), V(x, v, t))$, is, with its inverse Φ_t^{-1}, Holder continuous of exponent α, for any $\alpha < (1 - \sqrt{L_F/a^2})$.

In fact, since $z=(x, v)$, $z'=(x', v')$, and $T \geqslant t$,

$$|X(z, t)-X(z', t)| \leqslant |X(z, T)-X(z', T)|+\int_t^T ds\, |V(z, s)-V(z', s)|$$

$$|V(z, t)-V(z', t)| \leqslant |V(z, T)-V(z', T)| \tag{A.3}$$

$$+\int_t^T ds\, |F(X(z, s), s)-F(X(z', s), s))|$$

Let

$$\xi(t) \equiv \sqrt{L_F}\, |X(z, t)-X(z', t)|+|V(z, t)-V(z', t)|$$

and let us notice that, as $L_F>0$,

$$C_1\, |\Phi_t(z)-\Phi_t(z')| \leqslant |\xi(t)-\xi'(t)| \leqslant C_2\, |\Phi_t(z)-\Phi_t(z')| \tag{A.4}$$

By Eq. (A.3) and by the Lipschitz nature of F,

$$\xi(t) \leqslant \xi(T)+\sqrt{L_F} \int_t^T \xi(s)\, ds$$

By the Gronwall Lemma,

$$\xi(t) \leqslant \xi(T) \exp[\sqrt{L_F}\,(T-t) \tag{A.5}$$

From Eqs. (A.1) it follows that

$$|V(z, T)-V(z', T)| \leqslant |V(z, T)-v|+|V(z', T)-v'|+|z-z'|$$

$$\leqslant |z-z'|+2 \frac{\|F\|_{a, t_0}}{a} |z-z'| \tag{A.6}$$

and that

$$|X(z, T)-X(z', T)|$$

$$\leqslant |X(z, T)-(x+Tv)|+|X(z', T)-(x'+Tv')|+T|z-z'|$$

$$\leqslant 2 \frac{\|F\|_{a, t_0}}{a^2} |z-z'|+|z-z'|\, T. \tag{A.7}$$

Summing (A.6) and (A.7), we get

$$|\Phi_T(z)-\Phi_T(z')| \leqslant C(1+T)\, |z-z'|$$

where C is a constant depending on t, T, a, and L_F. Choosing

$$T = t + \frac{1}{a}\log\frac{1}{|z - z'|}$$

by (A.4), (A.5), we get

$$|\Phi_t(z) - \Phi_t(z')| \leqslant \frac{C}{C_1}\left|1 + \log\frac{1}{|z - z'|}\right| \cdot |z - z'|^{(1 - (\sqrt{L_F}/a))}$$

which proves Step 2. The Holder continuity of the inverse may be obtained analogously.

Step 3. By Step 2

$$f(x, v, t) \equiv f^*(\Phi_t)^{-1}(x, v) \tag{A.8}$$

is a weak solution of (2.5). Moreover, (2.6) is satisfied. In fact, as $t \to \infty$,

$$\begin{aligned}
&\|f(x, v, t) - f^*(x - vt, v)\|_{L_\infty(x, v)} \\
&\quad = \|f(x + vt, v, t) - f^*(x, v)\|_{L_\infty(x, v)} \\
&\quad = \|f^*(\Phi_t)^{-1}(x + vt, v) - f^*(x, v)\|_{L_\infty(x, v)} \\
&\quad = \|f^*(x + vt, v) - f^*(\Phi_t)(x, v)\|_{L_\infty(x, v)} \\
&\quad = \|f^*(x + vt, v) - f^*(X(x, v, t), V(x, v, t))\|_{L_\infty(x, v)} \to 0
\end{aligned}$$

where we have used the fact that Φ_t is bijective and that both $|X(x, v, t) - x + vt|$ and $|V(x, v, t) - v|$ vanish as $t \to \infty$.

Finally, since the vectorial field $(V, F(X, t))$ is divergence-free, then (see, for example, ref. 1), Φ_t is an area preserving map. Therefore, given a bounded function $\varphi(x, v)$, we have

$$\int_\Omega dy\, du\, \varphi(y, u)\, f(y, u, t) = \int_\Omega dx\, dv\, \varphi(X(x, v, t), V(x, v, t))\, f^*(x, v) \tag{A.9}$$

In particular

$$\mathscr{F}(F)(y, t) = \int_\Omega dx\, dv\, B(y - X(x, v, t))\, f^*(x, v) \tag{A.10}$$

Step 4. The spatial density $\rho(x, t)$ is bounded in the L_∞ norm: more precisely, as $t \geqslant t_0$,

$$\|\rho\|_\infty \leqslant 10 a_2 \tag{A.11}$$

In fact, for all $x \in S^1$, $\varepsilon > 0$, let us define

$$\rho_\varepsilon(x, t) \equiv \int_{B_\varepsilon(x)} dy\, \rho(y, t) = \int_{B_\varepsilon(x)} dy \int_{\mathbb{R}} dv\, f(y, v, t) \tag{A.12}$$

where $B_\varepsilon \subset S^1$ is the ball of radius ε centered in x. Let $(y', v') \equiv \Phi_t(y, v)$, and therefore $f(y, v, t) = f^*(y', v')$. By the fact that $f^*(y', v') \leqslant a_2/(1 + v'^4)$ and $|v - v'| \leqslant (\|F\|_{a, t_0}/a)\, e^{-at}$ [see (A.2)], it holds that

$$f(y, v, t) \leqslant a_2 \qquad \text{for} \quad |v'| < \frac{\|F\|_{a, t_0}}{a} e^{-at}$$

$$f(y, v, t) \leqslant \frac{a_2}{1 + |v| - (\|F\|_{a, t_0}/a)\, e^{-at}} \qquad \text{for} \quad |v'| \geqslant \frac{\|F\|_{a, t_0}}{a} e^{-at}$$

Then

$$\begin{aligned} \rho_\varepsilon(x, t) &\leqslant \int_{B_\varepsilon(x)} dy \int_{|v| < 2(A/a)\, e^{-at}} dv\, a_2 + \int_{B_\varepsilon(x)} dy \int_{|v| \geqslant 2(A/a)\, e^{-at}} dv \frac{a_2}{1 + (v'/2)^4} \\ &\leqslant 16\varepsilon a_2 + 4\varepsilon \frac{A a_2}{a} e^{-at} \leqslant 20 \varepsilon a_2 \end{aligned}$$

Dividing by 2ε and taking the limit for $\varepsilon \to 0$, we get (A.11).

Step 5. $\mathscr{F}(F)(x, t)$ is a Lipschitz function of x for any $t \geqslant t_0$, i.e.,

$$|\mathscr{F}(F)(x, t) - \mathscr{F}(F)(x', t)| \leqslant 24 a_2\, |x - x'|$$

In fact

$$\begin{aligned} |\mathscr{F}(F)(x, t) - \mathscr{F}(F)(x', t)| &\leqslant \int_{\mathbb{R}} dy\, |B(x - y) - B(x' - y)|\, \rho(y) \\ &\leqslant \int_{|y| \leqslant |x - x'|} dy\, |B(x - y) - B(x' - y)|\, \rho(y) \\ &\quad + \int_{|y| > |x - x'|} dy\, |B(x - y) - B(x' - y)|\, \rho(y) \\ &\leqslant 2\, |x - x'|\, 10 a_2 + |x - x'|\, 4 a_2 \end{aligned} \tag{A.13}$$

where we used (A.11), $\int \rho \leqslant 4 a_2$, and (2.3)–(2.4).

Step 6. Let us indicate with $X_i(x, v, t)$, $V_i(x, v, t)$, $i=1, 2$, the solutions of

$$\begin{aligned} X_i(x, v, t) &= x + vt + \int_t^\infty ds(s-t)\, F_i(X_i(s), s) \\ V_i(x, v, t) &= v - \int_t^\infty ds\, F_i(X_i(s), s). \end{aligned} \tag{A.14}$$

By the first of (A.14)

$$|X_1(x, v, t) - X_2(x, v, t)| \leqslant \int_t^\infty ds(s-t)\, |F_1(X_1(s), s) - F_2(X_1(s), s)|. \tag{A.15}$$

By (A.15)

$$\begin{aligned} |X_1(x, v, t) - X_2(x, v, t)| &\leqslant \int_t^\infty ds(s-t)(\|F_1\|_{a, t_0} + \|F_2\|_{a, t_0})\, e^{-at} \\ &\leqslant \frac{\|F_1\|_{a, t_0} + \|F_2\|_{a, t_0}}{a^2 e^{-at}} \end{aligned} \tag{A.16}$$

By Eq. (A.15) and by the Lipschitz nature of F_1, F_2 it holds that

$$\begin{aligned} &|X_1(x, v, t) - X_2(x, v, t)| \\ &\quad\leqslant \int_t^\infty ds(s-t)\, |F_1(X_1(s), s) - F_2(X_1(s), s)| \\ &\qquad+ \int_t^\infty ds(s-t)\, |F_2(X_1(s), s) - F_2(X_2(s), s)| \\ &\quad\leqslant \|F_1 - F_2\|_{a, t_0} \int_t^\infty ds(s-t)\, e^{-as} + \int_t^\infty ds(s-t)\, L_F\, |X_1(s) - X_2(s)|. \end{aligned} \tag{A.17}$$

Bootstrapping (A.17) starting from (A.16), we find

$$\begin{aligned} |X_1(x, v, t) - X_2(x, v, t)| &\leqslant \frac{2}{a^2 - L_F} \|F_1 - F_2\|_{a, t_0}\, e^{-at} \\ &\leqslant \frac{2}{a^2 - L_F} \|F_1 - F_2\|_{a, t_0} \end{aligned} \tag{A.18}$$

Step 7. Let us bound $\mathscr{F}(F_1)-\mathscr{F}(F_2)$. By (A.10), writing $|X_1(x, v, t) - X_2(x, v, t)|$ as ε for sake of simplicity, one has

$$
\begin{aligned}
&|\mathscr{F}(F_1)(x, v, t)-\mathscr{F}(F_2)(x, v, t)| \\
&\quad \leqslant \int dx\, dv\, f^*(x, v)\, |B(x-X_1(x, v, t))-B(x-X_2(x, v, t)| \\
&\quad \leqslant \int_{|X_1(x, v, t)|>\varepsilon} dx\, dv\, f^*(x, v)\, |B(x-X_1(x, v, t))-B(x-X_2(x, v, t))| \\
&\qquad +\int_{|X_1(x, v, t)|\leqslant\varepsilon} dx\, dv\, f^*(x, v)\, |B(x-X_1(x, v, t))-B(x-X_2(x, v, t)| \\
&\quad \leqslant L_B \int_{|X_1(x, v, t)|>\varepsilon} dx\, dv\, f^*(x, v)\, |X_1(x, v, t))-X_2(x, v, t)| \\
&\qquad +2b_\infty \int_{|X_1(x, v, t)|\leqslant\varepsilon} dx\, dv\, f^*(x, v). \qquad \text{(A.19)}
\end{aligned}
$$

where we have used (2.3) to estimate the first integral and (2.4) to estimate the second one.

Let I_1, I_2 be the first and the second integrals in the last of (A.19) respectively. It holds that

$$
\begin{aligned}
I_1 &\leqslant \frac{1}{2\pi}\int dx\, dv\, f^*(x, v)\, |X_1(x, v, t)-X_2(x, v, t)| \\
&\leqslant \frac{8a_2}{a^2-L_F}\, \|F_1-F_2\|_{a, t_0}
\end{aligned}
$$

where we have used (A.18) and

$$
\int dx\, dv\, f^*(x, v) \leqslant \int dx\, dv\, a_2/(1+v^4) \leqslant 16\pi a_2
$$

Moreover,

$$
\begin{aligned}
I_2 &= \int_{|y|\leqslant\varepsilon} dy\, \rho(y, t) \leqslant \frac{40a_2}{a^2-L_F}\, \|F_1-F_2\|_{a, t_0}\left(1+\frac{\|F_1\|_{a, t_0}}{a}\, e^{-at}\right) \\
&\leqslant \frac{80a_2}{a^2-L_F}\, \|F_1-F_2\|_{a, t_0}
\end{aligned}
$$

where we used (A.18).

Summing I_1, I_2, we get (3.7).

Step 8. Finally, let us prove (3.8). With $F=0$, we have $f(x, v, t)=f^*(x-vt, v)$. Therefore

$$\partial_x \mathscr{F}(0)(x, t)=\int dv f^*(x-vt, v)-\rho_0$$

and

$$\hat{\mathscr{F}}(0)(k, t)=\hat{f}^*(k, kt) \qquad \text{for} \quad k\neq 0$$
$$\hat{\mathscr{F}}(0)(0, t)=0$$

where $\hat{\mathscr{F}}(0)(k, t)$ is the Fourier transform of $\mathscr{F}(0)(x, t)$ with respect to x, and

$$\hat{f}^*(a, b)\equiv\frac{1}{2\pi}\int dx\, dv\, e^{-iax-ibv} f^*(x, v)$$

is the Fourier transform of $f^*(x, v)$ with respect to x and v. Therefore

$$|\mathscr{F}(0)(x, t)|\leqslant\sum_{k\neq 0}\frac{1}{|k|}|\hat{f}^*(k, kt)|\leqslant\sum_{k\neq 0}\frac{1}{|k|}\frac{1}{1+k^2}a_2 e^{-ak^2 t}$$
$$\leqslant a_1 a_2 e^{-at}\sum_{k\neq 0}\frac{1}{|k|}\frac{1}{1+k^2}a_2\leqslant 4a_1 a_2 e^{-at}$$

that is,

$$\|F\|_{a, t_0}\leqslant 4a_1 a_2$$

Finally, noticing that $\|\mathscr{F}(F)\|_{a, t_0}\leqslant\|\mathscr{F}(0)\|_{a, t_0}+\|\mathscr{F}(F)-\mathscr{F}(0)\|_{a, t_0}$, and by Step 7, we get (3.5).

Proof of Theorem 3.4.

Let us consider the solution (Φ_t, E, f) of (2.7)–(2.9) constructed in Theorem 3.2. In particular, let

$$\Phi_t(x, v)\equiv(X(x, v, t), V(x, v, t))$$
$$(\Phi_t)^{-1}(X, V)\equiv(x(X, V, t), v(X, V, t))$$

Step 1. For any $b\in(0, a)$ there exists a positive constant c_1 such that, for any $(x, v)\in\Omega$, for any $t\geqslant t_0$

$$|X(x, v, t)-tV(x, v, t)-x|+|V-v|\leqslant c_1 e^{-bt} \tag{A.20}$$

In fact

$$X(x, v, t) - tV(x, v, t) = x - \int_{\infty}^{t} ds\, sE(X(x, v, s), s)$$

$$V(x, v, t) = v + \int_{\infty}^{t} ds\, E(X(x, v, s), s)$$

Therefore, since $|E(x, t)| \leqslant 8a_1 a_2 e^{-at}$ (see Theorem 3.2),

$$|X(x, v, t) - tV(x, v, t) - x| \leqslant 8a_1 a_2 \int_{t}^{\infty} ds\, se^{-as} = 8a_1 a_2 \left(\frac{1}{a^2} + \frac{t}{a}\right) e^{-at}$$

$$|V(x, v, t) - v| \leqslant 8a_1 a_2 \int_{t}^{\infty} ds\, e^{-as} = \frac{1}{a} e^{-at}$$

and therefore (A.20) holds.

Step 2. For any $b \in (0, a)$ there exists a positive constant c_2 such that, as $t \geqslant t_0$,

$$\|\rho - \rho_0\|_{L_\infty(x)} \leqslant c_2 e^{-bt} \tag{A.21}$$

In fact,

$$\begin{aligned} |\rho(X, t) - \rho_0| &= \left| \int_{\mathbb{R}} f(X, V, T)\, dV - \rho_0 \right| \\ &= \left| \int_{\mathbb{R}} f^*(\Phi_t)^{-1} (X, V)\, dV - \rho_0 \right| \\ &\leqslant \left| \int_{\mathbb{R}} |f^*(\Phi_t)^{-1} (X, V) - f^*(X - Vt, V)|\, dV \right| \\ &\quad + \left| \int_{\mathbb{R}} f^*(X - Vt, V)\, dV - \rho_0 \right| \end{aligned} \tag{A.22}$$

The first integral in the last of (A.22), call it $I_1(X)$, may be bounded in the following way:

$$\begin{aligned} I_1(X) &= \left| \int dV [f^*(x(X, V, t), v(X, V, t)) - f^*(X - Vt, V)] \right| \\ &\leqslant \int dV \frac{1}{1 + (\min(v(X, V, t), V))^2} |x(X, V, t) - X - tV| \end{aligned} \tag{A.23}$$

where we have used (3.15). By the fact that $|v - V|$ is bounded by const $\cdot\, e^{-at_0}$, we get (A.21). The second integral in the last of (A.22), call it $I_2(X)$, may be explicitly evaluated using the Fourier transform:

$$\hat{I}_2(k, t) = \hat{f}^*(k, kt) \qquad \text{for} \quad k \neq 0$$
$$\hat{I}_2(k, t)(0, t) = 0$$

where $\hat{I}_2$ is the Fourier transform of $I_2(X, t)$ with respect to X, and $\hat{f}^*(a, b)$ is the Fourier transform of $f^*(x, v)$ with respect to x and v. Therefore,

$$\|I_2(X, t)\|_{L_\infty(X)} \leqslant \sum_{k \neq 0} |\hat{f}^*(k, kt)| \leqslant \sum_{k \neq 0} \frac{1}{1 + k^2} a_2 e^{-ak^2 t}$$
$$\leqslant a_1 a_2 e^{-at} \sum_{k \neq 0} \frac{1}{1 + k^2} a_2 \leqslant 4 a_1 a_2 e^{-at} \qquad \text{(A.24)}$$

By (A.23) and (A.24), we get (A.21).

Step 3. By (A.21) and $\partial_x E = \rho(x, t) - \rho_0$ we get

$$|E(x, t) - E(x', t)| \leqslant c_2 e^{-at} |x - x'| \qquad \text{(A.25)}$$

Now the proof follows by standard methods. In fact, by (A.25), it follows easily that Φ_t and its inverse $(\Phi_t)^{-1}$ are Lipschitz and therefore that f and ρ are Lipschitz. From this we get that $\partial_x E$ is Lipschitz, and this, in particular, implies that E is C^1, and by (A.25), that $|\partial_x E(x, t)| \leqslant c_2 e^{-bt}$. This implies that Φ_t and its inverse $(\Phi_t)^{-1}$ are C_1. Finally, since f^* is C^1, $f \equiv f^* \circ (\Phi_t)^{-1}$ is C^1 and the * solution is classical.

Remark. When this work was complete, the referee pointed out in his report that two recent papers[29, 30] were concerned with the same problem considered here. In the first one, the author claims the electric field decays algebraically, such as $1/t$ and not exponentially as $t \to \infty$. He conjectures this fact by considering the function $U(a, b)$ which is the asymptotic velocity of a particle which at time 0 is in (a, b). He notices in particular that a generic observable at time t may be computed evaluating integrals like $\int da\,db\,f_0(a, b)\, e^{iU(a, b)t}$ a part for terms which do not oscillate. Then he notices that if the gradient of U vanishes in some point then, by stationary phase arguments, the integral above should vanish as $t \to \infty$ in an algebraic way. Finally he gives an argument for which the gradient of U should vanish in many points.

The error in this argument is due to the fact that the gradient of U cannot vanish. In fact, in this paper, we have constructed the transformation $X(a, b)$, $U(a, b)$ which links the initial datum (a, b) to its asymptotic behaviour $X + Ut$, U. This transformation is canonical, therefore its Jacobian is 1, and therefore it cannot happen that $\partial_a U = \partial_b U = 0$.

For what concerns the result in [30], the author shows, numerically, that the electric field, after some initial damping, definitively oscillates around some constant value. We agree with the conjecture that VPE's solutions can converge to a steady solution which in general is not homogeneous. We note that this is not in contrast with out result as in this paper we construct particular initial conditions for which the electric field asymptotically decays.

ACKNOWLEDGMENTS

We thank C. Marchioro and P. Negrini for helpful conversations and, in particular, D. Benedetto for Theorem 3.4 and M. Pulvirenti for a critical reading of the manuscript. Furthermore, we thank a referee for his interesting comments. This work was partially supported by MURST (Ministero dell'Universitá e della Ricerca Scientifica e Tecnologica) and CNR (Consiglio Nazionale delle Richerche–Gruppo Nazionale per la Fisica Matematica).

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